

ON THE INTEGRAL VALUES OF A CURIOUS RECURRENCE

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ABSTRACT. We discuss a problem initially thought for the Mathematical Olympiad but which has several interpretations. The recurrence sequences involved in this problem may be generalized to recurrence sequences related to a much larger set of diophantine equations.

The purpose of this note is to comment on a problem shortlisted for the Romanian Mathematical Olympiad 2010 (see [1, Shortlisted Problems for the 61th NMO, No. 16]).

Problem Let x_0, x_1, x_2, \dots be the sequence defined by

$$(1) \quad \begin{aligned} x_0 &= 1 \\ x_{n+1} &= 1 + \frac{n}{x_n}, \end{aligned} \quad \forall n \geq 0.$$

What are the values of n for which x_n is an integer?

The author of this problem is Gheorghe Iurea, but his solution does not appear in the above quoted booklet.

We have found this problem of interest, not only in itself, but also because *a posteriori* it may be dealt with in different ways, each of which involves mathematical arguments of various nature. For instance, the second solution below uses a linear differential equation of the second order, which admits a solution which is a well-known function in combinatorics.

It may be that the arguments extend to cover a whole bunch of problems of a similar sort.

Let us now go back to the above sequence. Note that its first few values are

$$(2) \quad 1, 1, 2, 2, \frac{5}{2}, \frac{13}{5}, \frac{38}{13}, \frac{58}{19}, \frac{191}{58}, \frac{655}{191}, \dots$$

We can iterate the recursive formula (1) to obtain $x_{n+2} = 1 + \frac{n+1}{1 + \frac{n}{x_n}} = \frac{(n+2)x_n + n}{x_n + n}$; in general, x_{n+k} may be expressed in two different ways: first, by a kind of continued fraction involving x_n and the integers in $\{n, \dots, n+k-1\}$, second, by a linear fractional transformation in x_n , namely $x_{n+k} = M_{n,k}(x_n) = \frac{\alpha_{n,k}x_n + \beta_{n,k}}{\gamma_{n,k}x_n + \delta_{n,k}}$, for suitable integer coefficients depending on n, k ; here $M_{n,k}$ is associated to the matrix $\begin{pmatrix} \alpha_{n,k} & \beta_{n,k} \\ \gamma_{n,k} & \delta_{n,k} \end{pmatrix}$.

These matrices satisfy the recurrence

$$M_{n,k+1} = \begin{pmatrix} 1 & n+k \\ 1 & 0 \end{pmatrix} M_{n,k}.$$

The alluded continued fraction has not bounded length and is not periodic, and it seems not easy to find a simple formula for the general term x_n or for the matrices $M_{n,k}$. Also, the standard tools using congruences do not seem to lead directly to a

solution of the stated question. Nevertheless we shall see a number of methods to answer it.

FIRST SOLUTION

We now present a solution which turns out to be essentially the same of Gheorghe's, which he kindly sent us.

If we define $f_n(x) = 1 + \frac{n}{x}$, we see that $x_{n+1} = f_n(x_n)$, so that one is led to study the dynamics of the sequence of functions f_n ; we note that in this solution the *arithmetic* comes into play only at the end, whereas one starts just by studying the dynamics from the real variable viewpoint (rather than a variable in \mathbb{Q}).

Let us call $y_n = \frac{1+\sqrt{4n+1}}{2}$ the (positive) fixed point of f_n . Plainly we have that if $x < y_n$ then $f_n(x) > y_n$ and vice versa.

We can prove by induction that

Lemma 1. *For every $n \geq 4$ we have*

$$(3) \quad y_{n-1} = \frac{1 + \sqrt{4n-3}}{2} < x_n < \frac{1 + \sqrt{4n+1}}{2} = y_n.$$

Proof. By a direct computation, we have $\frac{1+\sqrt{13}}{2} < \frac{5}{2} < \frac{1+\sqrt{17}}{2}$, which establishes the basis of the induction. Assuming (3) holds for n , by the previous remark we have that $y_n < x_{n+1}$, so we need only to prove that

$$1 + \frac{n}{x_n} < y_{n+1}.$$

By the inductive hypothesis, it is enough to show that

$$\begin{aligned} 1 + \frac{n}{y_{n-1}} &< y_{n+1}, \quad i.e., \\ 1 + \frac{2n}{1 + \sqrt{4n-3}} &< \frac{1 + \sqrt{4n+5}}{2}, \end{aligned}$$

which is an elementary, though tedious, computation. \square

Remark 1. For the values of n smaller than 4, we have $x_3 = y_2 = x_2 = 2$ and $x_1 = y_0 = x_0 = 1$.

Let us now assume that x_n is an integer for some $n \geq 4$. From the lemma we have

$$\begin{aligned} \frac{1 + \sqrt{4n-3}}{2} &< x_n < \frac{1 + \sqrt{4n+1}}{2}, \\ \sqrt{4n-3} &< 2x_n - 1 < \sqrt{4n+1}, \\ 4n-3 &< (2x_n-1)^2 < 4n+1. \end{aligned}$$

However the last inequalities are inconsistent modulo 4.

Therefore we conclude that the only integral values of the sequence are x_0, \dots, x_3 .

Essentially the same solution may be reached by a slightly different approach.

The same conclusion as before can be reached if we show that $n-1 < x_n^2 - x_n < n$ for $n \geq 4$.

We argue by induction. The inequalities are verified by direct inspection for $n = 4$, since $3 < \frac{25}{4} - \frac{5}{2} < 4$.

Now let $a_n = x_n^2 - x_n$, and assume that the inequalities hold up to n . We may write a_{n+1} as

$$a_{n+1} = x_{n+1}^2 - x_{n+1} = x_{n+1}(x_{n+1} - 1) = \left(1 + \frac{n}{x_n}\right) \frac{n}{x_n} = \frac{n(x_n + n)}{x_n^2}.$$

By the induction hypothesis we have:

$$x_n^2 < x_n + n \implies a_{n+1} > n \frac{(x_n + n)}{x_n + n} = n$$

and

$$x_n^2 > x_n + n - 1 \implies a_{n+1} < n \frac{x_n + n}{x_n + n - 1} < n + 1$$

since $x_n > 1$.

SECOND SOLUTION

To study the sequence (x_n) from an arithmetic point of view we define two integer sequences $(a_n), (b_n)$ by the recurrences

$$\begin{aligned} a_0 &= 1 \\ b_0 &= 1 \\ (4) \quad a_{n+1} &= a_n + nb_n, & \forall n \geq 0 \\ (5) \quad b_{n+1} &= a_n, & \forall n \geq 0. \end{aligned}$$

Comparing (4) and (5) with (1) we see immediately that they satisfy $x_n = \frac{a_n}{b_n}$, so a_n, b_n are the numerator and denominator respectively in some fractional representation of x_n ; however a_n, b_n *a priori* need not be coprime, so the said fraction can be possibly simplified.

We also see that b_n may be eliminated from the recurrence to get

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ (6) \quad a_{n+2} &= a_{n+1} + (n+1)a_n, & \forall n \geq 0, \end{aligned}$$

with $x_n = \frac{a_n}{a_{n-1}}$.

Let us define $d_n = \gcd(a_n, a_{n-1})$; d_n tells us how much the reduced denominator of x_n differs from a_{n-1} . So, to obtain a lower bound for said denominator, we need a lower bound for a_{n-1} and an upper bound for d_n .

Remark 2. By the recurrence (6) we see that $d_{n+1} | a_{n+2}$, and so $d_{n+1} | d_{n+2}$; this will be helpful in establishing an upper bound for d_n .

A lower bound for a_n is easily obtained as in the following lemma.

Lemma 2. *For every $n \geq 0$ we have $a_n \geq \sqrt{n!}$.*

Proof. We argue by induction on $n \geq 0$. We check that $a_0 = 1 = \sqrt{0!}$, $a_1 = 1 = \sqrt{1!}$, and assuming the bound for a_n and a_{n+1} we get

$$\begin{aligned} a_{n+2} &= a_{n+1} + (n+1)a_n \geq \sqrt{(n+1)!} + (n+1)\sqrt{n!} = \\ &= \sqrt{(n+2)!} \frac{1 + \sqrt{n+1}}{\sqrt{n+2}} = \sqrt{(n+2)!} \sqrt{1 + \frac{2\sqrt{n+1}}{n+2}} \geq \sqrt{(n+2)!}. \quad \square \end{aligned}$$

To get an upper bound for d_n we introduce the exponential generating function of the sequence $(a_n)_{n \in \mathbb{N}}$, namely

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

We consider $F(x)$ merely as a formal power series, although one could prove that it converges for every complex x .

From the recurrence on (a_n) we can obtain a differential equation for F ; in fact, we can multiply (6) by $\frac{x^n}{n!}$ and sum it for $n \geq 0$; since clearly $F'(x) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n$ and $F''(x) = \sum_{n=0}^{\infty} \frac{a_{n+2}}{n!} x^n$, we obtain that F satisfies the conditions

$$(7) \quad \begin{cases} F(0) &= 1 \\ F'(0) &= 1 \\ F''(x) &= (x+1)F'(x) + F(x). \end{cases}$$

The Cauchy problem (7) may be solved (in the ring of formal power series) to get

$$F(x) = e^{x + \frac{x^2}{2}},$$

and we can use this explicit form to get a formula for a_n . In fact

$$(8) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n &= e^{x + \frac{x^2}{2}} = e^x e^{\frac{x^2}{2}} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{s=0}^{\infty} \frac{x^{2s}}{2^s s!} \\ \frac{a_n}{n!} &= \sum_{2s+m=n} \frac{1}{2^s s!} \frac{1}{m!} \\ a_n &= \sum_{2s \leq n} \frac{n!}{2^s s! (n-2s)!} = \sum_{2s \leq n} \binom{n}{2s} (2s-1)!!, \end{aligned}$$

where the *semifactorial* $(2s-1)!!$ denotes as usual the product $(2s-1) \cdot (2s-3) \cdots 3 \cdot 1$ and is defined to be 1 for $s = 0$.

Remark 3. We note that these explicit formulas enable us to improve on Lemma 2. Indeed, using Stirling's formula, one may deduce that the 'correct' order of magnitude of $\frac{a_n}{\sqrt{n!}}$ is roughly $\exp(\sqrt{n})$. A corresponding upper bound may be also obtained directly by induction, using the recurrence for a_n .

We can now use the preceding formula to prove the following lemma.

Lemma 3. *Let p be an odd prime. If $p|n$, then $a_n \equiv 1 \pmod{p}$.*

Proof. Let p be an odd prime dividing n .

If $p < 2s$, we have that $p|(2s-1)!!$, as p itself is one of the factors in the defining product of $(2s-1)!!$.

If $0 < 2s < p$ the binomial $\binom{n}{2s}$ is divisible by p , as the p factor in n is not cancelled by $(2s)!$.

So we have that in formula (8) only the term with $s = 0$ is not divisible by p , whence

$$a_n = \sum_{2s \leq n} \binom{n}{2s} (2s-1)!! \equiv 1 \pmod{p}. \quad \square$$

Applying this lemma we get the following property of d_n .

Corollary 4. *For every $n \geq 1$, d_n is a power of 2.*

Proof. If an odd prime p divides d_m for some $m \geq 1$, then, by Remark 2, p divides d_n (and hence a_n) for all $n \geq m$, so also for $n = pm$. But this is not possible because $a_{pm} \equiv 1 \pmod{p}$ by Lemma 3. \square

We are now ready to prove an upper bound for d_n , which will follow by using again the exponential generating function $F(x)$.

Proposition 5. *For every $n \geq 1$ we have that $d_n \leq 2^{n-1}$.*

Proof. We have the following identities concerning the above generating function $F(x)$:

$$\left(\sum_{m=0}^{\infty} a_m \frac{x^m}{m!} \right) \left(\sum_{r=0}^{\infty} (-1)^r a_r \frac{x^r}{r!} \right) = F(x)F(-x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Comparing the coefficients of x^{2n} for any $n \geq 1$, we obtain

$$(9) \quad \sum_{m+r=2n} (-1)^r \binom{2n}{m} a_m a_r = \frac{(2n)!}{n!} = 2^n \cdot (2n-1)!!.$$

Now, as observed in Remark 3 above, d_{n+1} divides any a_m with $m \geq n$, so it divides the left-hand side of (9), and we know from the Corollary 4 that it is a power of 2, therefore $d_{n+1} \leq 2^n$ for $n \geq 1$; of course $d_1 = 1 = 2^0$. \square

To get the conclusion, denote by D_n the *reduced* denominator of x_n . We have:

$$D_n \geq \frac{a_{n-1}}{d_n} \geq \frac{\sqrt{(n-1)!}}{2^{n-1}} \quad \forall n \geq 1.$$

It is easily seen that

$$\sqrt{(n-1)!} > 2^{n-1} \quad \forall n \geq 10,$$

so we are left to inspect the values of x_n with $0 \leq n \leq 9$, which are exactly the values listed in (2).

Remark 4. It is probably worth noting that the exponential generating function $F(x)$ is widely known in the literature, and for instance it can be interpreted as the exponential generating functions of the number a_n of involutions of the symmetric group \mathcal{S}_n (see for instance [2, Thm 3.16]).

FURTHER OBSERVATIONS

We can actually say much more about the numbers d_n .

Proposition 6. *If we define*

$$e_n = \begin{cases} k & \text{if } n = 4k \\ k & \text{if } n = 4k + 1 \\ k + 1 & \text{if } n = 4k + 2 \\ k + 2 & \text{if } n = 4k + 3, \end{cases}$$

then, for every $n \geq 0$, the exact power of 2 dividing a_n equals 2^{e_n} .

Proof. Letting $q_n := a_n/2^{e_n}$, we are reduced to prove that q_n is an odd integer for $n \geq 0$. We need the following lemma.

Lemma 7. *For every $n \geq 2$ we have*

$$a_{n+6} = 2(n^2 + 9n + 19)a_{n+2} - n(n-1)(n+2)(n+5)a_{n-2}.$$

Proof. Indeed, since the sequence (a_n) verifies a linear recurrence of the *second* order, any *three* sequences of the shape $(a_n), (a_{n+r}), (a_{n+s})$ are linearly related by an equation with coefficients which are polynomials in n ; they may be found by easy elimination. Presently, we are interested in the case $r = 4, s = 8$, where this elimination is hidden in the following explicit calculations:

$$\begin{aligned}
a_{n+6} &= a_{n+5} + (n+5)a_{n+4} = (n+6)a_{n+4} + (n+4)a_{n+3} \\
&= (2n+10)a_{n+3} + (n+6)(n+3)a_{n+2} \\
&= (n^2 + 11n + 28)a_{n+2} + 2(n+5)(n+2)a_{n+1} \\
&= 2(n^2 + 9n + 19)a_{n+2} - (n+2)(n+5)a_{n+2} + 2(n+2)(n+5)a_{n+1} \\
&= 2(n^2 + 9n + 19)a_{n+2} + (n+2)(n+5)a_{n+1} - (n+2)(n+5)(n+1)a_n \\
&= 2(n^2 + 9n + 19)a_{n+2} - n(n+2)(n+5)a_n + n(n+2)(n+5)a_{n-1} \\
&= 2(n^2 + 9n + 19)a_{n+2} - n(n-1)(n+2)(n+5)a_{n-2}. \quad \square
\end{aligned}$$

Remark 5. It is worth noticing that the relation with *polynomial* coefficients that we have obtained is ‘monic’, in the sense that the coefficient of a_{n+6} is 1. This feature, which is for us important, is not *a priori* guaranteed for a linear recurrence with polynomial coefficients, and appears to us as a piece of good luck.

Having proved the lemma, if we divide the relation in this last lemma by $2^{e_{n+6}}$ we obtain the recurrence

$$q_{n+6} = (n^2 + 9n + 19)q_{n+2} - \frac{n(n-1)(n+2)(n+5)}{4}q_{n-2}.$$

Observe that $n^2 + 9n + 19$ is odd for every n , while $\frac{n(n-1)(n+2)(n+5)}{4}$ is an even integer for every n : indeed, for n even (resp. n odd), the product $n(n+2)$ (resp. $(n-1)(n+5)$) is divisible by 8.

So, after checking by inspection that

$$q_0 = 1, q_1 = 1, q_2 = 1, q_3 = 1, q_4 = 5, q_5 = 13, q_6 = 19, q_7 = 29$$

are all odd integers, we obtain by induction that the q_n are all odd integers. \square

In view of the previous definitions and by Corollary 4, we can now compute d_n by a ‘closed’ formula:

Proposition 8.

$$d_n = \begin{cases} 2^k & \text{if } n = 4k \\ 2^k & \text{if } n = 4k + 1 \\ 2^k & \text{if } n = 4k + 2 \\ 2^{k+1} & \text{if } n = 4k + 3. \end{cases}$$

As a final remark, we observe that Proposition 8 easily implies that:

- (a) The estimate $d_n \leq 2^{\frac{n+1}{4}}$ holds for all $n > 0$, which improves on the lower bound of the reduced denominator D_n to

$$D_n \geq \frac{\sqrt{(n-1)!}}{2^{\frac{n+1}{4}}} \quad \forall n \geq 1,$$

and says directly that $D_n > 1$ for $n \geq 4$;

- (b) the denominator D_n ($n > 0$) is even if and only if $n \equiv 0 \pmod{4}$, while the corresponding numerator is even if and only if $n \equiv 2, 3 \pmod{4}$.

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